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The influence of failure criteria on strength prediction of ceramic components

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Abstract

The effect of failure criteria on the computation of short-time strength distributions of ceramic components, based on the evaluation of test results obtained from typical experimental setups, is investigated. Three different sensitivity indicators are defined to quantify the influence of (a) multiaxial failure criteria, (b) the cracks' behaviour under compressive loading, and (c) inexact measurement of the Weibull modulus of the strength distribution. Numerical values for different states of stress as well as for typical experimental setups are given and preliminary statements are derived.

Keywords: Failure analysis; Fracture; Multiaxial failure criteria; Strength

1. Strength prediction of ceramic components

Besides being brittle and having low tensile strength, ceramic components show large scatter of short-time strength and life expectancy. By the principles of linear-elastic fracture mechanics, this can be explained by a flaw population with random orientation and statistical distributed size.[1–4] Hence, a component's strength depends on its volume, the distribution of stresses, the assumed type, size and density of flaws—and a failure criterion. All these influences are summarized in a variable called 'effective volume':

$$V_{\mathrm{eff}} = \frac{1}{4\pi}\int_V\int_0^{2\pi}\int_0^{\pi}\left[\frac{\sigma_{\mathrm{Ieq}}(x,y,z,\varphi,\theta)}{\sigma^*}\right]^m \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}V. \tag{1}$$

σ^* specifies the component's loading (e.g. the highest tensile stress or an applied external pressure). The exponent m derives from the size distribution of the assumed flaw population and is found again in the failure probability (Eq. 4). $\sigma_{\mathrm{Ieq}}(x,y,z,\varphi,\theta)$ is a comparative (equivalent) stress for a crack with orientation φ,θ located at x, y, z under mixed-mode load. It is calculated using a failure criterion such as the criterion after Richard[5]:

$$\sigma_{\mathrm{Ieq}} = \frac{1}{2}\left[\sigma_{\mathrm{n}} + 4\sqrt{\sigma_{\mathrm{n}}^2 + \left(\alpha_{\mathrm{I}}\frac{Y_{\mathrm{II}}}{Y_{\mathrm{I}}}\tau_{\mathrm{eff}}\right)^2}\right]. \tag{2}$$

Y_{I} and Y_{II} are factors describing the geometry of the cracks. The parameter α_{I} allows the adaption to test results and sets the sensitivity against shearing stress, respectively mode II and mode III loading (mode I: tension normal to crack plane, mode II: shear loading normal to crack front, mode III: shear loading in direction of crack front), on the crack. Due to this adjustability, the criterion after Richard is later used to define sensitivity indicators.

σ_{n} is the stress normal to the crack plane. In case of negative σ_{n}, the crack is under compressive loading and three assumptions concerning the effect of shear stress (τ_{eff}) are possible: (a) no failure occurs, (b) failure occurs only as a result of shear loading ($\sigma_{\mathrm{n}}=0$, $\tau_{\mathrm{eff}}=\tau$), (c) failure due to shear loading decreased by frictional force, which leads to the relation of Alpa[6]:

$$\tau_{\mathrm{eff}} = \max(0; |\tau| - |\mu\sigma_{\mathrm{n}}|) \text{ for } \sigma_{\mathrm{n}} < 0. \tag{3}$$

Likewise, this relation is used to define sensitivity indicators, because by variation of the parameter μ it

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can be fitted to test results and covers the cases (a) and (b).

As a whole, the effective volume can be described as the volume of tensile-test rods, which show the same distribution of strength as the original component, when comparing the tensile stress of the rod with σ^*. The test rod's flaw population has to be the same as the component's, but with flaws orientated perpendicular to tensile stress, so that failure criteria are effectless. Surface flaws respectively flaw populations related to the component's surface are considered by an analog approach.

The strength distribution respectively the failure probability P_f under a given load σ^* finally reads:

$$P_f(\sigma^*) = 1 - \exp\left[-\left(\frac{\sigma^*}{\sigma_0^*}\right)^m\right]. \tag{4}$$

This is a Weibull distribution with the two parameters m (Weibull modulus) and σ_0^*, a so-called 'characteristic' reference stress. Calculation of a component's short-time strength distribution (subscript 'A') requires a conversion of the Weibull parameters obtained from tests with specimens (typically four-point bending specimens) (subscript 'B'):

$$\sigma_0^*,_{\mathrm{A}} = \sigma_0^*,_{\mathrm{B}} \cdot \left(\frac{V_{\mathrm{eff,B}}}{V_{\mathrm{eff,A}}}\right)^{\frac{1}{m}}. \tag{5}$$

This holds under the assumption, that in both cases the same type of flaw population leads to failure. Also, an applicable failure criterion has to be used.

But how to quantify the effect of the choice of failure criteria? This question is explored in the following sections.

2. Definition of sensitivity indicators

The influence of failure criteria finally is reflected in the parameter σ_0^* [Eq. (4)], which is calculated using the effective volume of the specimen [Eq. (5)]. Hence, the effective volume [Eq. (1)] is used to define numbers to quantify the effect of failure criterion on short-term strength prediction of ceramic components. These numbers will be termed 'sensitivity indicators'.

Two different sensitivity indicators for failure criteria (I_F) are introduced: the first one describes the sensitivity against shearing stress, the second one describes the cracks' behaviour under compressive load. An additional indicator I_m concerning the Weibull modulus m is suggested. Because of their adjustability, the failure criterion after Richard (parameter α_I) in combination with the criterion after Alpa (parameter μ) are used to define the following sensitivity indicators:

$$I_{F1} = \frac{\Delta\left(V_{\mathrm{eff}}^{1/m}\right)}{\overline{V_{\mathrm{eff}}^{1/m}} \cdot \Delta\alpha_I}\Bigg|_{\alpha_{I1}=0.95;\mu=0;m=10}^{\alpha_{I2}=1.05} \tag{6}$$

$$I_{F2} = \frac{\Delta\left(V_{\mathrm{eff}}^{1/m}\right)}{\overline{V_{\mathrm{eff}}^{1/m}} \cdot \Delta\mu}\Bigg|_{\mu_1=0;\alpha_I=1;m=10}^{\mu_2=1} \tag{7}$$

In addition, the effect of inexact measurement of the Weibull modulus m is quantified by the sensitivity indicator I_m:

$$I_m = 100 \cdot \frac{\Delta q}{\overline{q} \cdot \Delta m}\Bigg|_{m_1=9.5;\alpha_I=1;\mu=0}^{m_2=10.5} \quad \text{with} \quad q = \left(\frac{V_{\mathrm{eff}}}{V}\right)^{1/m} \tag{8}$$

The indicators are defined by two values of effective volume (V_{eff1} and V_{eff2} respectively their difference and average). They are calculated for different parameters α_I and μ of the criteria after Richard and Alpa under the assumption of circular cracks ($Y_{II}/Y_I = 1.117$ at the location on the crack front with the highest equivalent stress, when using a failure criterion which is insensitive against mode III loading). In case of I_m, the Weibull modulus m is varied instead of one of the parameters. All these indicators do not depend on the absolute level of stresses and stay constant upon rescaling of the component. To ensure this, the additional consideration of the component's volume V is necessary in the definition of I_m.

I_{F1} is an approximation of the normalized partial derivative of $V_{\mathrm{eff}}^{1/m}$ with respect to α_I (with $\alpha_I = 1$, $\mu = 0$ and $m = 10$). $\mu = 0$ was chosen to avoid numerical problems when all three principal stresses are compressive. Otherwise, in these cases both effective volumes can be zero due to decrement of shear loading by frictional forces. For α_I in the criterion after Richard usually a range from 0.5 to 1.3 is given. Hence the definitions use $\alpha_I \approx 1$. Also, the Weibull modulus varies from 6 (alumina) to 18 (zirconia, silicon nitride up to 20), so $m = 10$ is chosen.

I_{F2} is the mean normalized partial derivative of $V_{\mathrm{eff}}^{1/m}$ with respect to μ within limits $0 \leqslant \mu \leqslant 1$ (with $\alpha_I = 1$ and $m = 10$). The sign in Eq. 7 keeps the range of possible values above zero. I_m represents the normalized change of $(V_{\mathrm{eff}}/V)^{1/m}$ when increasing m from 9.5 to 10.5.

3. Numerical values

Since the indicators depend only on the ratio of the principal stresses to each other and not on the stresses' absolute values, Figs. 1 and 2 show the indicators' values for spatial constant states of stress. In these

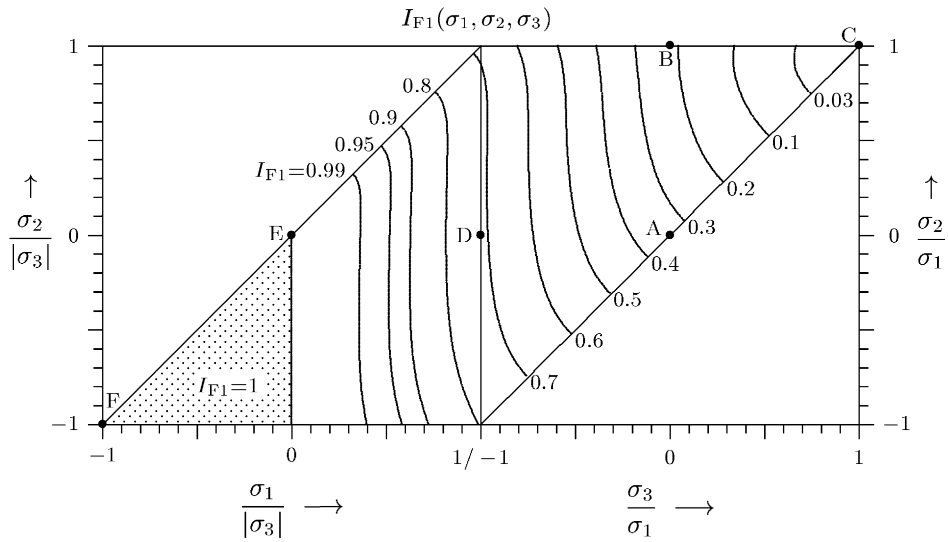

Fig. 1. Values of I_{F1} for all possible states of stress (see Table 2 for special cases A–F).

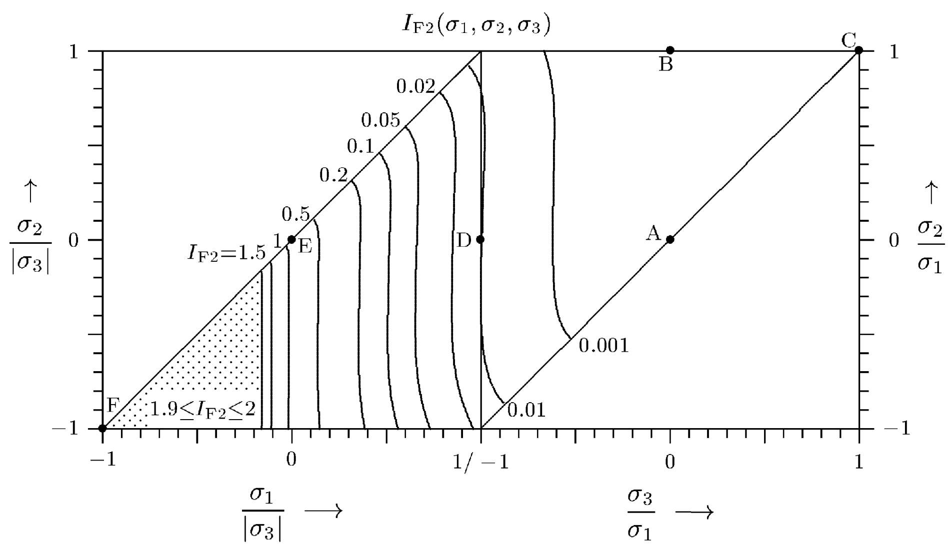

Fig. 2. Values of I_{F2} for all possible states of stress.

figures, tensile stresses are represented by values above zero. With $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$, these diagrams cover all possible states of stress by normalizing stresses either to σ_1 (right half) or to $|\sigma_3|$ (left half, in case of $|\sigma_3| > \sigma_1$). Values have been calculated numerically for a dense, grid-like array of points over the diagram's area. The codomains of I_{F1} and I_{F2} read as follows:

$$0 \leqslant I_{F1} \leqslant 1 \text{ and } 0 \leqslant I_{F2} \leqslant 2.$$

For I_m no range of values can be given, because a variation of m changes the weighting of the local states of stress in Eq. (1). Since sensitivity indicators represent derivatives of V_{eff}, this leads to an additional influence of gradients of stresses, hence higher or lower values than shown in Fig. 3 can occur (see Table 2 for some examples). For spatial constant states of stress the codomain of I_m ranges from 0 to approx. 1.38.

3.1. Relation between I_{F1} and I_{F2}

The figures also show a nearly parallel course of isolines for I_{F1} and I_{F2}. This illustrates a direct relation between the two indicators—at least for the spatial constant states of stress. Plotting I_{F2} versus I_{F1} for all possible states of stress leads to the relation shown in Fig. 4. The maximum deviation of the numerical calculated values of I_{F2} from the plotted line is less than 0.006 for $I_{F1} \leqslant 0.995$. In case of complex stress distributions, the relation between the two indicators is less clear as the results for an axial piston in Table 1 show.

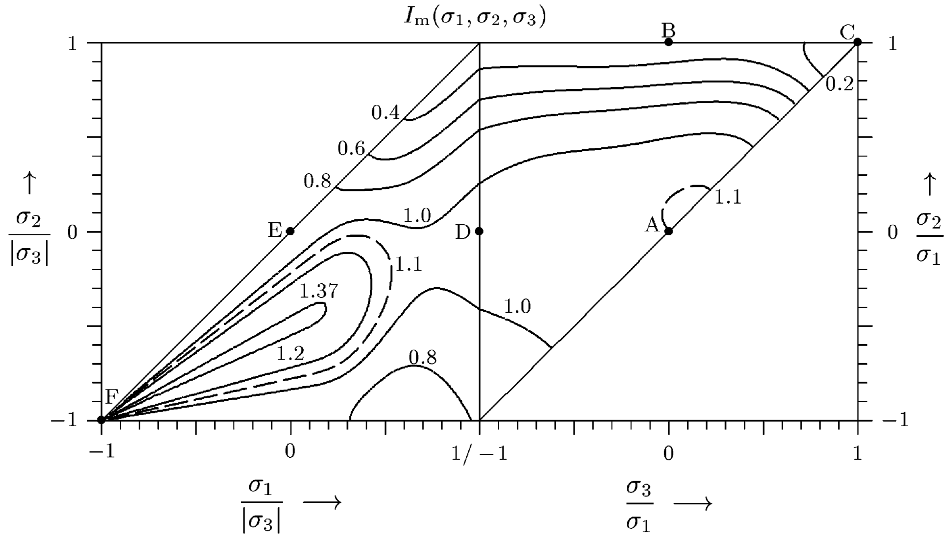

Fig. 3. Values of I_m for all possible states of stress.

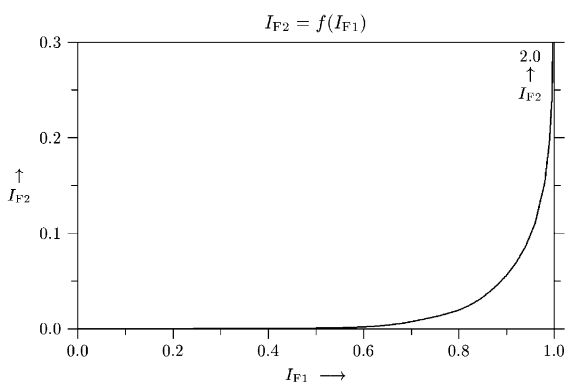

Fig. 4. Relation between I_{F1} and I_{F2} for constant spatial stress distributions.

Table 1
Values of sensitivity indicators for selected experimental setups respectively types of specimens

Experimental setup/ type of specimen	I_{F1}	I_{F2}	I_m
Tensile test	0.34	0.00	1.10
4-Point bending test	0.34	0.00	4.07
Ring-on-ring test	0.22	0.00	3.72
Cold-spin test	0.30	0.00	4.64
Brazilian-disk test	1.00	0.94	5.62
Torsion (tube[a])	0.72	0.01	1.73
Ring[b], internal pressure	0.34...0.89	0.00...0.05	1.10...7.42
Axial piston pump[c]			
Piston	0.52	0.02	5.51
Bush	0.69	0.10	4.73
Sliding block	0.27	0.00	7.64

[a] Ratio of diameters: 1:2.
[b] Ratio of diameters from approx. 1:1 to 1:20.
[c] Example, specific pump under specific operating conditions.

Table 2
Special cases (A–F) of stress states and corresponding sensitivity indicators

Case	Description	Principal Stresses	I_{F1}	I_{F2}	I_m
A	Uniaxial tensile stress	$\sigma_1>0;\ \sigma_2=\sigma_3=0$	0.34	<0.001	1.10
B	Equibiaxial tensile stress	$\sigma_1=\sigma_2>0;\ \sigma_3=0$	0.22	<0.001	0.33
C	Equitriaxial tensile stress	$\sigma_1=\sigma_2=\sigma_3>0$	0.00	0.00	0.00
D	Torsion	$\sigma_3=-\sigma_1;\ \sigma_2=0$	0.72	0.01	1.05
E	Uniaxial compressive stress	$\sigma_3<0;\ \sigma_1=\sigma_2=0$	1.00	0.85	0.85
F	Hydrostatic compressive stress	$\sigma_1=\sigma_2=\sigma_3<0$	1.00	2.00	–

For states of stress with mainly tensional character (right half of Fig. 2), the value of I_{F2} is nearly constant, whereas I_{F1} shows a wide range of values. As tensile stresses are of particular relevance to failure of ceramics, the first effect indicator I_{F1} is certainly the more important one, when rating specimens regarding their applicability to reveal failure criteria. Only in cases of anticipated failure due to compressive stress the second indicator may be used.

3.2. Alternative definitions

Alternative definitions of sensitivity indicators are possible. For example, the effect of failure criteria on strength predictions can be defined as the ratio of $V_{\text{eff}}^{1/m}$ under assumption of two different failure criteria instead of varying the parameter α_I of the criterion after Richard. Also definitions can use different parameter values and can assume other Weibull moduli m. Finally, there is an infinite number of conceivable definitions.

A comparison of I_{F1} with two arbitrary defined ratios of $V_{eff}^{1/m}$ is illustrated in Fig. 5 (again for all possible states of stress with spatial constant stresses).

The curves represent upper and lower limits of dependance on I_{F1}. The examples read (Examples 1, 2):

Example 1. V_{eff1} was calculated using the maximum energy release rate criterion,[5] V_{eff2}: criterion of normal stress. In both cases the Weibull modulus was set to $m=15$ and failure under compressive loading was not allowed.

Example 2. V_{eff1}: criterion of normal stress, V_{eff2}: criterion after Richard ($\alpha_I=1$). Weibull modulus in both cases $m=6$, failure under compressive load due to shear loading was allowed (no frictional forces).

The figure shows, that these two alternative definitions are related roughly in a monotonic way to I_{F1}. A clear distinction of two states of stress with a wide difference in I_{F1} can be found again when using them instead of I_{F1}. This ensures the applicability of the sensitivity indicator at least for the given cases with other failure criteria and Weibull moduli than used in the indicator's definition.

3.3. Sensitivity indicators for typical experimental setups

With the aid of finite element method, stress distributions of specimens in typical experimental setups (see Fig. 6) were investigated and the relevant sensitivity indicators were calculated. For this purpose, a numerical tool was developed, which evaluates result-sets of FEM simulations to obtain the effective volumes of the specimens. Table 1 shows sensitivity indicators I_{F1}, I_{F2} and I_m for the displayed setups respectively specimens.

The ring-on-ring test (concentric ring test) offers with it's biaxial stress state (cp. Table 2) the lowest value of I_{F1}. As seen in Fig. 1, point B, a further reduction is only possible by adding a third principal tensile stress. On the other hand, the Brazilian disk test shows high values for I_{F1} and I_{F2}, which implies a combined effect of the chosen failure criterion and cracks' behaviour under compressive loading.

An experimental setup, which covers at least a comparatively wide range of I_{F1}-values and low values of I_{F2}, is the testing of concentric rings under internal pressure and a gradient of pressure on it's faces, which arises from gap flow between the ring and a fixation plate (as shown in Fig. 6, bottom right). Rings of different diameter ratios can be used in the same test rig, thus different values of I_{F1} can be attained. Table 2 shows the range of possible values for the defined indicators. These ranges are of theoretical character, because maximum pressure and aspects of manufacturing limit the applicable diameter ratios. A setup designed by the author allows a range of $0.41 \leqslant I_{F1} \leqslant 0.67$. The corresponding diameter ratios are 0.81 and 0.5.

4. Conclusion

The effective volume, a basic value to predict strength of ceramic components, is used to define two sensitivity indicators I_{F1} and I_{F2}. They quantify the effect of failure criteria on strength prediction and allow distinction of the particular influences of the multiaxial failure criterion (e.g. maximum energy release rate criterion, criterion after Richard) and of an additional criterion for the cracks' behaviour under compressive load. A third indicator (I_m), associated with the Weibull modulus of the strength distribution, is also defined. Numerical values for spatial constant states of stress, for typical

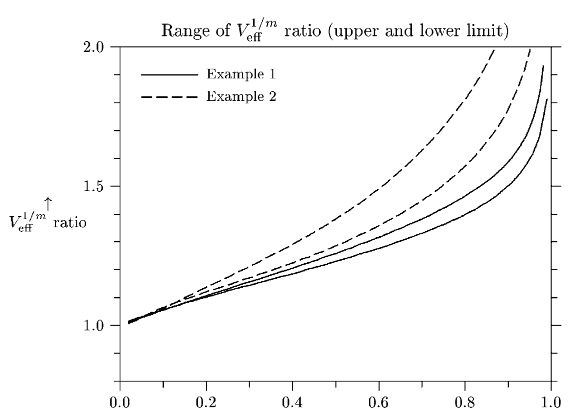

Fig. 5. Demonstration of applicability of the I_{F1} indicator in cases of different Weibull moduli and different failure criteria than used for the indicator's definition.

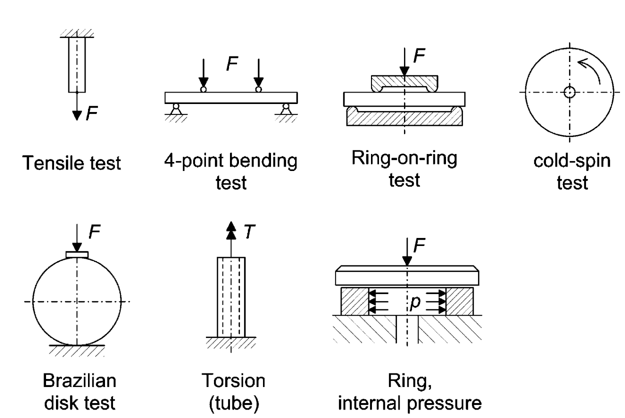

Fig. 6. Typical experimental setups and types of specimen for short-term strength tests and lifetime tests of ceramic materials.

experimental setups and for real components of an axial piston pump are given. Comparability of indicator I_{F1} with alternative definitions is shown by means of two examples.

Finally, the following preliminary statement is derived: to extract an applicable failure criterion for a specific material, test results obtained from a test with low I_{F1} should be compared with results from a test with high I_{F1}. The Brazilian disk test was already suggested for the determination of multiaxiality criteria.[7] It shows the highest possible value for I_{F1}. But at the same time I_{F2} is comparatively high, which means an additional interference of cracks' behaviour under compressive load. Hence, the Brazilian disk test seems to be applicable only for strength prediction of components with mainly compressive states of stress ($|\sigma_3| >> \sigma_1$). A pure torsional loading, for example, also shows a high value for I_{F1}, but I_{F2} is nearly zero. This raises the question if test results obtained exclusively from a combination like ring-on-ring/Brazilian disk are sufficient for all cases of stress fields. For a more comprehensive description of the material's behaviour under multiaxial loading three requirements are suggested, which consist of the following tests: (a) both indicators, I_{F1} and I_{F2}, are of low value; (b) I_{F1} is as high as possible while I_{F2} is still low; (c) both indicators are high. In all three cases I_m should be as low as possible to minimize errors due to inexact measurement of the Weibull modulus m. Testing concentric rings of varying diameter ratios under internal pressure is suggested to cover a medium range of I_{F1}. Corresponding test series are in process.

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